BED-FORMS AND INSTABILITY OF SAND BEDS

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- When the flow over a sedimentary bed exceeds the threshold value, the bed does not remain stable but takes different features known as *bed-forms*
- The shape and size of the bed-forms depends on the flow characteristics
- Bed-forms significantly influence on various flow parameters

Types of Bed-Forms

- For the purpose of the classification of bed-forms, three flow regimes are distinguished according to the flow Froude number F [= $U/(gD_h)^{0.5}$, where U = depth-averaged flow velocity; g = acceleration due to gravity; D_h = hydraulic depth, A/T; A = flow area; and T = top width of flow]

➢ Lower regime for $F < 1$; e.g. ripples, ripples on dunes and dunes

➢ Transition for $F \approx 1$; e.g. washout dunes

➢ Upper regime for $F > 1$; e.g. plane beds, antidunes, chutes and pools

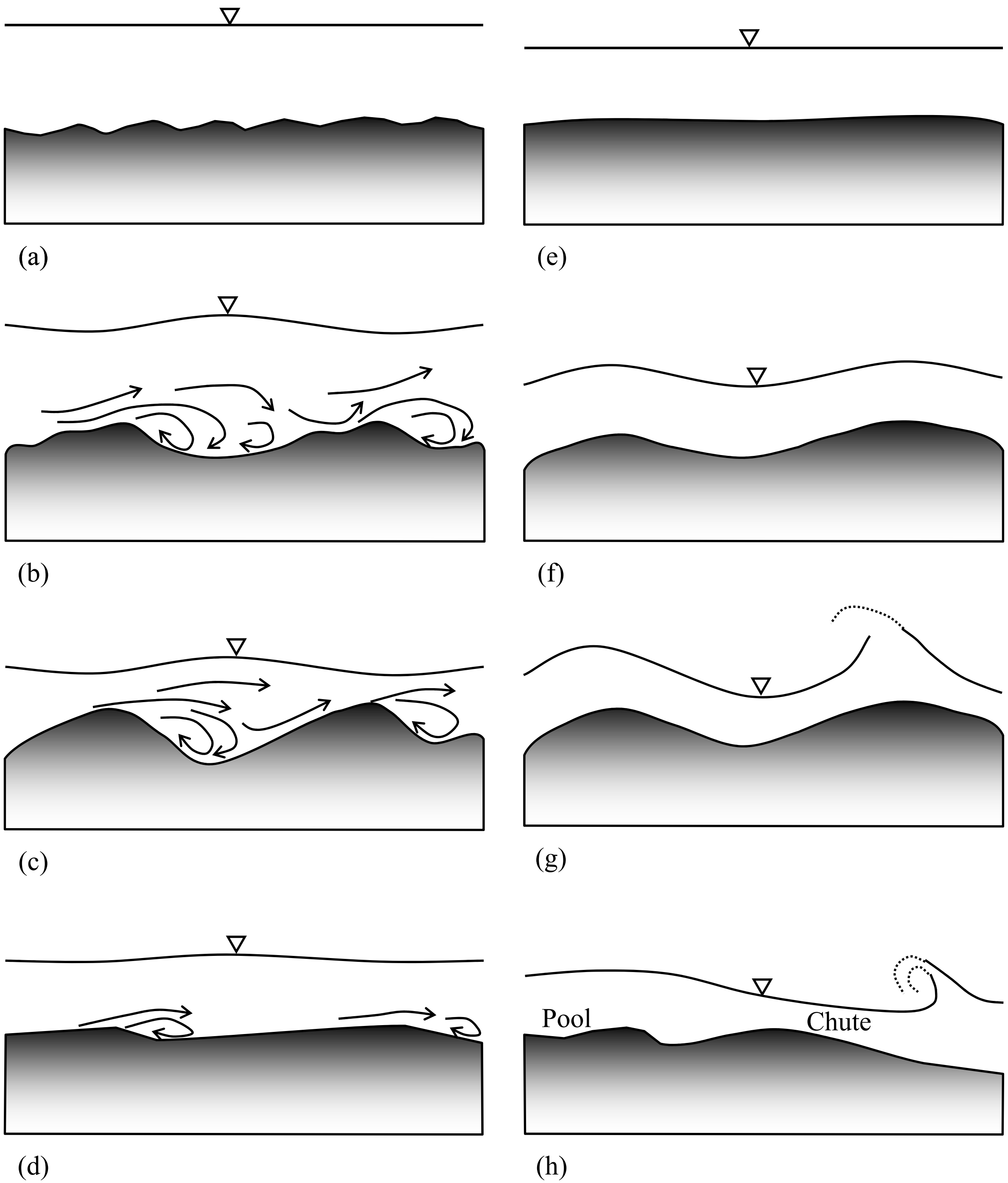
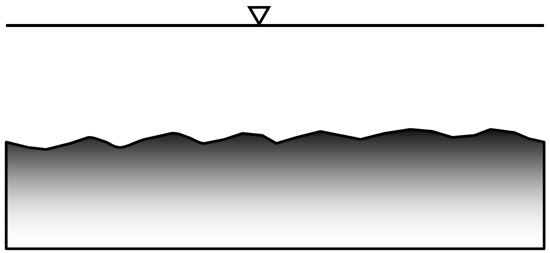


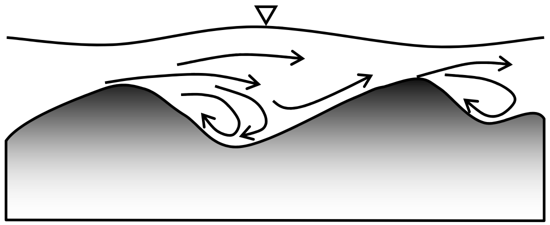
Fig. 5.1 Schematic of bed-forms: (a) Ripples; (b) ripples on dunes; (c) dunes; (d) transition or washed out dunes; (e) plane bed; (f) antidune standing waves; (g) antidune breaking wave; and (h) chutes and pools

Ripples



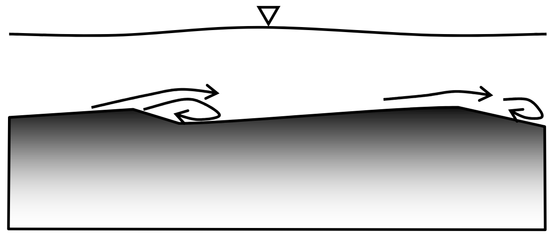
- Small triangular sand waves with long gradual upstream slope (approximately 6°) and short steep downstream slope (approximately 32°) are called *ripples*
- In case of fine sediments (d_{50} < 0.7 mm), ripples are formed, while coarse sediments usually form dunes
- Ripples are formed if viscous sub-layer is present when the threshold shear stress is just suppressed
- Length of the ripples depends on the sediment size and the flow velocity, but is essentially independent of the flow depth
- Ripples may be superposed upon the upstream side of dunes

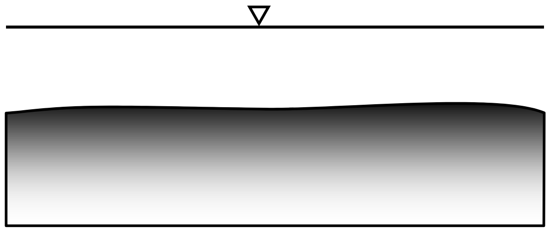
Dunes



- *Dunes* are the bed-forms larger than ripples, whose profile is out of phase with the free surface profile
- The streamwise profile of a dune is roughly triangular with a mild upstream slope and a downstream slope approximately equal to the angle of repose
- Flow separation that occurs at the crest of a dune reattaches in the trough, so that bottom rollers are formed on the lee side of a dune
- Near the zone of reattachment, the sediment particles are transported by the turbulence, even when the local bed shear stress is below its threshold value

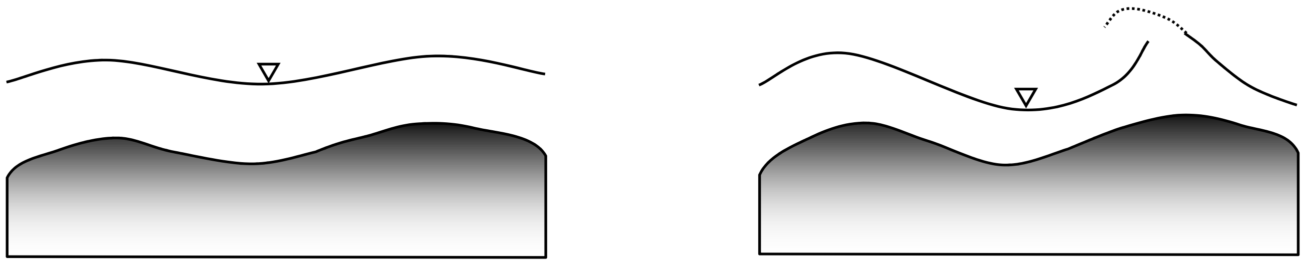
Transition and Plane Bed





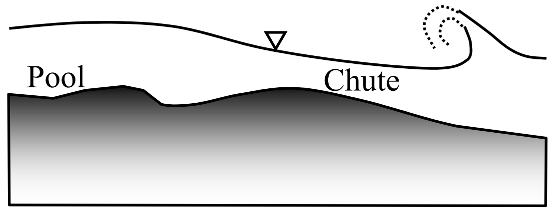
- For increased stream power (product of the velocity and the bed shear stress), the dunes tend to wash out and they become progressively longer and flatter and finally disappear
- This change from dunes to plane bed means a rather drastic reduction of both hydraulic resistance and flow depth

Antidunes



- Bed and the free surface profiles are in phase
- The streamwise bed profile is nearly sinusoidal and so is free surface profile, but usually with much larger amplitude
- At lower Froude numbers, antidunes appear as *standing sand wave*
- At higher Froude numbers, the sand wave may grow becoming unstable and breaking in the upstream direction
- If the latter occurs, the antidunes are destroyed, the bed becomes flat, and the formation of antidunes starts all over again

Chutes and Pools



- Extremely strong antidunes actively lead to *chutes* and *pools* flow, which occur at relatively large slopes with high flow velocities and sediment concentrations
- They consist of large elongated heaps of sediment
- Shooting flow on the heaps of sediment runs into a pool where the flow is generally tranquil

Instability of Sand Beds

Objectives

- To develop a new theory of turbulent shear flow over a wavy bed using the Reynolds averaged Navier-Stokes (RANS) and the time-averaged continuity equations addressing:

(1) The characteristics of free surface profiles over stable sinusoidal sand beds

(2) The instability criterion of erodible beds leading to the formation of sand waves

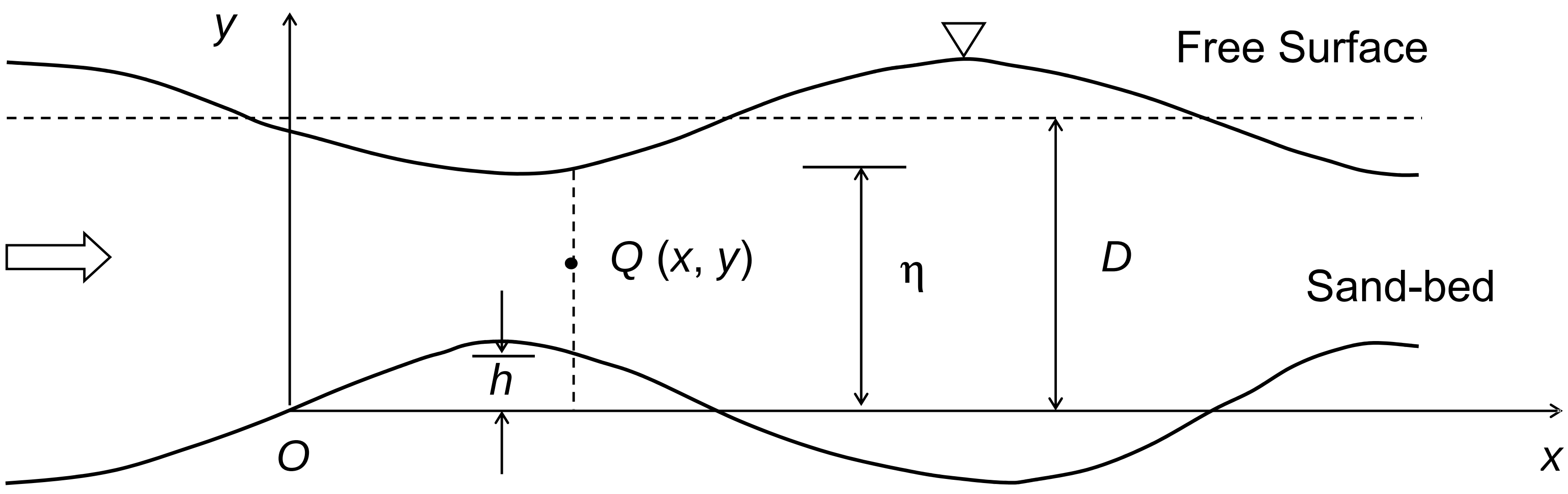


Figure 1 Definition sketch of flow over an undulating sand-bed

- For erodible sand-bed, the bed elevation of sand wave $h = h(x, t)$
- The elevation of wavy free surface profile $\eta = \eta(x, t)$
- The mean flow depth D is constant

- The instantaneous velocity components (u, v) at a point $Q(x, y)$ is split into time-averaged part $(\bar{u}, \bar{v})$ and fluctuation part (u', v') as

$$u(x,y,t) = \bar{u}(x,y,t) + u'(x,y,t)\ , \quad v(x,y,t) = \bar{v}(x,y,t) + v'(x,y,t) \qquad (1)$$

The continuity equations for $(\bar{u}, \bar{v})$ and (u', v') are

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \quad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \qquad (2)$$

- The exact RANS equations of 2D turbulent flow are

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{\partial \overline{P}}{\partial x} + \frac{\partial \tau}{\partial y} + \upsilon\frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial(\overline{u'^2})}{\partial x} \quad (3a)$$

$$\frac{\partial \overline{v}}{\partial t} + \overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} = -\frac{\partial \overline{P}}{\partial y} + \frac{\partial \tau}{\partial x} + \upsilon\frac{\partial^2 \overline{v}}{\partial x^2} - \frac{\partial(\overline{v'^2})}{\partial y} - g \quad (3b)$$

where $\overline{P}(x, y, t)$ is the time-averaged hydrostatic pressure relative to mass density of fluid ρ, $\tau(x, y, t)$ is $-\overline{u'v'}$ that is the Reynolds shear stress relative ρ, υ is the kinematic viscosity of fluid and g is the gravitational acceleration

Eqs. 2 - 3b form an undetermined system, since there are six dependent parameters ($\overline{u}$, $\overline{v}$, u', v', $\overline{P}$ and τ)

Turbulence Assumptions

- The gradients of the Reynolds stresses along *x* are nearly zero. Thus

$$\partial\tau/\partial x \approx 0, \quad \partial(\overline{u'^2})/\partial x \approx 0, \quad \partial(\overline{v'^2})/\partial x \approx 0 \qquad (4)$$

The second assumption is on the law of variation of $\overline{u}$ with *y* in turbulent flow

- A single averaged distribution of $\overline{u}$ is assumed following the 1/*p*-th power law, where *p* is usually taken as 7 for the turbulent flow over a rigid boundary. Thus

$$\overline{u} = U_0(x,t)\left(\frac{y-h}{\eta-h}\right)^{1/p} \qquad (5)$$

where U_0 is the maximum velocity at $y = \eta$

- A theoretical approximation by the power law may be sought in turbulent stresses that dominate the viscous stresses in Eqs. 3a and 3b. Thus

$$\left|\upsilon\, \partial^2\overline{u}/\partial y^2\right| << \left|\partial\tau/\partial y\right|, \quad \left|\upsilon\, \partial^2\overline{v}/\partial x^2\right| << \left|\partial\tau/\partial x\right| \approx 0 \qquad (6)$$

- For a slowly varying variable, $\zeta = (y - h)^{1/p}$, where $p > 1$. Since $y - h = \zeta^p$ and $dy = p\zeta^{p-1}d\zeta$, the first condition in Eq. 6 becomes

$$\left|\partial^2 \overline{u} / \partial \zeta^2\right| << (p\zeta^{p-1} / \upsilon)\left|\partial \tau / \partial \zeta\right| \quad (7)$$

so that

$$\left|\partial^2 \overline{u} / \partial \zeta^2\right| \leq \varepsilon \ll (p\delta^{p-1} / \upsilon)\min\{\left|\partial \tau / \partial \zeta\right|\} \quad \text{with} \quad \zeta \geq \delta \quad (8)$$

where ε and $\delta \geq 0$ are small constants

- The left hand side of the above inequality implies that $-\varepsilon \leq \partial^2\overline{u}/\partial\zeta^2 \leq \varepsilon$, whose appropriate solution is

$$\overline{u} = A(x,t) + B(x,t)\zeta + \theta\varepsilon\zeta^2 \quad \text{with } \zeta \geq \delta \text{ and } -0.5 \leq \theta \leq 0.5 \quad (9)$$

where θ is an uncertainty function. As $\overline{u} \to 0$ for $\zeta \to 0$, $A(x, t) = 0$

- Setting $B(x, t) = U_0(x, t)/(\eta - h)^{1/p}$ and dropping the uncertain small term containing $\theta\varepsilon$, one gets $\overline{u}$ in the form of Eq. 5, that remains valid for $\zeta \to 0$ or $y \to h$

- In Eq. 5, the term U_0 represents the maximum velocity at a flow section, which can be related to the depth-averaged velocity $U(x, t)$ as

$$U(x,t) = \frac{1}{\eta - h}\int_h^\eta \overline{u}\,dy = \frac{p}{1+p}U_0(x,t) \tag{10}$$

- Thus, using Eq. 10, Eq. 5 can be written as a function of U as

$$\overline{u} = \frac{1+p}{p}U(x,t)\left(\frac{y-h}{\eta - h}\right)^{1/p} \tag{11}$$

- The continuity equation, Eq. 2, then yields

$$\overline{v} = -(\eta - h)\frac{\partial U}{\partial x}\left(\frac{y-h}{\eta - h}\right)^{(1+p)/p} \tag{12}$$

Flow Assumptions

- The free surface profile possesses a curvature with insignificant streamwise gradient. It implies that as $|\partial h/\partial x| \approx 0$, $|\partial \eta/\partial x| \approx 0$

  By Eq. 2, the advective vertical acceleration is

$$\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} = \bar{u}\frac{\partial \bar{v}}{\partial x} - \bar{v}\frac{\partial \bar{u}}{\partial x} = \bar{u}^2\frac{\partial(\tan\psi)}{\partial x} \approx \bar{u}^2\kappa \qquad (13)$$

  where $\tan\psi$ is the slope of a streamline through the point Q (x, y) and κ is the curvature of the streamline through the point Q, such that $\kappa(h) \approx \partial^2 h/\partial x^2$ and $\kappa(\eta) \approx \partial^2 \eta/\partial x^2$

- Following the Boussinesq theory, one can assume a linear variation of κ between the curvatures $\kappa(h)$ and $\kappa(\eta)$ at levels h and η (that is $h \le y \le \eta$), respectively, so that

$$\kappa = \kappa(h) + [\kappa(\eta) - \kappa(h)]\frac{y-h}{\eta-h} \qquad (14)$$

With this value of κ in Eq. 13 and $\overline{u}$ given by Eq. 5, Eq. 3b is integrated with respect to *y* and the resulting equation is

$$\overline{P} = \overline{P}_0 + g(\eta - y) - U^2(\eta - h)\left(\frac{1+p}{p}\right)^2 \left\{ \frac{p}{2+p}\kappa(h)\left[\left(\frac{y-h}{\eta-h}\right)^{(2+p)/p} - 1\right] + \frac{p}{2(p+1)}[\kappa(\eta) - \kappa(h)]\left[\left(\frac{y-h}{\eta-h}\right)^{(2+p)/p} - 1\right]\right\} - \overline{v'^2} \quad (15)$$

where $\overline{P}_0$ is the value of $\overline{P}$ at $y = \eta$. The above equation yields $\partial \overline{P}/\partial x$, noting that the contribution of v'^2 is negligible due to negligible variations of turbulence stresses, as given in Eq. 4

The expression for $\partial \overline{P}/\partial x$ is used in the momentum equation, Eq. 3a

Depth-averaged Equations

- Taking the depth-averaged continuity equation, Eq. 2, and using Eq. 10, one can write

$$\frac{D\eta}{Dt}-\frac{Dh}{Dt}=\bar{v}\Big|_h^\eta=-\int_h^\eta\frac{\partial\bar{u}}{\partial x}dy=-\frac{\partial}{\partial x}[(\eta-h)U]+\bar{u}(x,\eta,t)\frac{\partial\eta}{\partial x}-\bar{u}(x,h,t)\frac{\partial h}{\partial x} \qquad (16)$$

where $D(\cdot)/Dt = \partial(\cdot)/\partial t + \bar{u}\,\partial(\cdot)/\partial x$. Eq. 16 thus reduces to

$$\frac{\partial}{\partial t}(\eta-h)+\frac{\partial}{\partial x}[(\eta-h)U]=0 \qquad (17)$$

- For depth averaging Eq. 3a, from Eq. 15 one gets

$$\int_h^\eta\frac{\partial\bar{P}}{\partial x}dy=g(\eta-h)\frac{\partial\eta}{\partial x}+\gamma\frac{\partial}{\partial x}\left[U^2(\eta-h)^2\kappa(\eta)+\frac{p}{2(p+1)}\kappa(h)\right] \qquad (18)$$

where $\gamma = (p+1)^2/[p(3p+2)]$.

- Similarly, for the advective acceleration by partially integrating the third term of the left hand side of Eq. 3a using Eqs. 2 and 11, one gets

$$\int_h^\eta \left(\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} \right) dy = \frac{\partial}{\partial t}[(\eta - h)U] + \frac{\partial}{\partial x}\int_h^\eta \overline{u}^2 dy$$

$$= \frac{\partial}{\partial t}[(\eta - h)U] + \sigma \frac{\partial}{\partial x}[(\eta - h)U] \qquad (19)$$

where $\sigma = (p + 1)2/[p(p + 2)]$

- Using Eq. 4, the integration of Eq. 3a with respect to y, yields

$$\frac{\partial}{\partial t}[(\eta - h)U] + \sigma\frac{\partial}{\partial x}[(\eta - h)U] + \gamma\frac{\partial}{\partial x}\left\{(\eta - h)^2 U^2\left[\kappa(\eta) + \frac{p}{2(p+1)}\kappa(h)\right]\right\}$$
$$+ g(\eta - h)\frac{\partial \eta}{\partial x} + gn^2\frac{U^2}{(\eta - h)^3} = 0 \qquad (20)$$

where n is the Manning roughness coefficient

- Reynolds stress $\tau(y)$ vanishes at $y = h$ and η. The bed shear stress τ_0 is represented in Eq. 20 by applying the Manning equation locally as $\rho u_\tau^2 = \tau_0 = \rho g n^2 U^2 (\eta - h)^{1/3}$; where u_τ is the shear velocity at distance x

- Differentiating of Eq. 20, one obtains an alternative form of Eq. 20

$$\frac{\partial U}{\partial t} + (2\sigma - 1)U\frac{\partial U}{\partial x} + (\sigma - 1)\frac{U^2}{\eta - h}\cdot\frac{\partial}{\partial x}(\eta - h) + \gamma(\eta - h)U^2\left[\frac{\partial \kappa(\eta)}{\partial x} + \frac{p}{2(p+1)}\frac{\partial \kappa(h)}{\partial x}\right]$$

$$+2\gamma U\left[\kappa(\eta) + \frac{p}{2(p+1)}\kappa(h)\right]\frac{\partial}{\partial x}[(\eta - h)U] + g\frac{\partial \eta}{\partial x} + gn^2\frac{U^2}{(\eta - h)^{4/3}} = 0 \qquad (21)$$

- Eq. 20 or 21 can be viewed a generalization of the Saint Venant equation, considering 1/*p*-th power law of velocity and the curvature of streamlines

- For application of Eq. 21, $p \approx 7$, yielding $\sigma \approx 1$ and $\gamma \approx 2/5$

- In these approximations, the coefficient of the second term of the left hand side in Eq. 21 is unity, while the third term becomes negligible

Free Surface Profiles Over Stable Undulating Sand-Beds

- For steady flow over an undulating sand-bed, the flow depth *h* and the depth-averaged velocity *U* are invariant of *t*; and continuity equation yields

$$(\eta - h)U = q \tag{22}$$

where *q* is the discharge per unit width. Eliminating *U* from Eq. 21 with the aid of Eq. 22, yields the differential equation of the wavy free surface profile as

$$\frac{d^3\eta}{dx^3} - \frac{5}{2q^2}\cdot\frac{q^2 - g(\eta - h)^3}{(\eta - h)^2}\cdot\frac{d\eta}{dx} + \frac{7}{16}\cdot\frac{d^3h}{dx^3} + \frac{5}{2(\eta - h)^2}\cdot\frac{dh}{dx} + \frac{5}{2}\cdot\frac{gn^2}{(\eta - h)^{7/3}} = 0 \tag{23}$$

- If the bed has a sinusoidal form as $h = a \sin(kx)$, where *a* is the amplitude and *k* is the wave number, Eq. 23 in nondimensional form is given by

$$\frac{d^3\hat{\eta}}{d\hat{x}^3} - \frac{5}{2\beta^2} \cdot \frac{F^2 - [1+\alpha(\hat{\eta}-\sin\hat{x})]}{F^2[1+\alpha(\hat{\eta}-\sin\hat{x})]^2} \cdot \frac{d\hat{\eta}}{d\hat{x}} - \frac{7}{16}\cos\hat{x} + \frac{5}{2\beta^2} \cdot \frac{\cos\hat{x}}{[1+\alpha(\hat{\eta}-\sin\hat{x})]^2}$$

$$+ \frac{5}{2} \cdot \frac{\varphi}{\alpha\beta^3} \cdot \frac{1}{[1+\alpha(\hat{\eta}-\sin\hat{x})]^{7/3}} = 0 \qquad (24)$$

where

α is the nondimensional amplitude of bed-form (= a/D)

β is the wave number with respect to mean flow depth (= kD)

F is the Froude number [= $q/(gD^3)^{0.5}$]

φ is the bed characteristic parameter (= $n^2g/D^{1/3}$)

$\hat{x}$ is the nondimensional horizontal distance (= kx)

$\hat{\eta}$ is the nondimensional vertical distance [= $(\eta - D)/(\alpha D)$]

- A typical numerical experiment was conducted for α = 0.1, β = 13, F = 0.2 and φ = 4×10-3. Eq. 24 was solved by the Runge-Kutta method

- A satisfactory solution was obtained for the initial values of $\hat{\eta}$ = 0.8, $d\hat{\eta}/d\hat{x}$ = -0.71 and $d^2\hat{\eta}/d\hat{x}^2$ = –0.002 at the origin $\hat{x}$ = 0

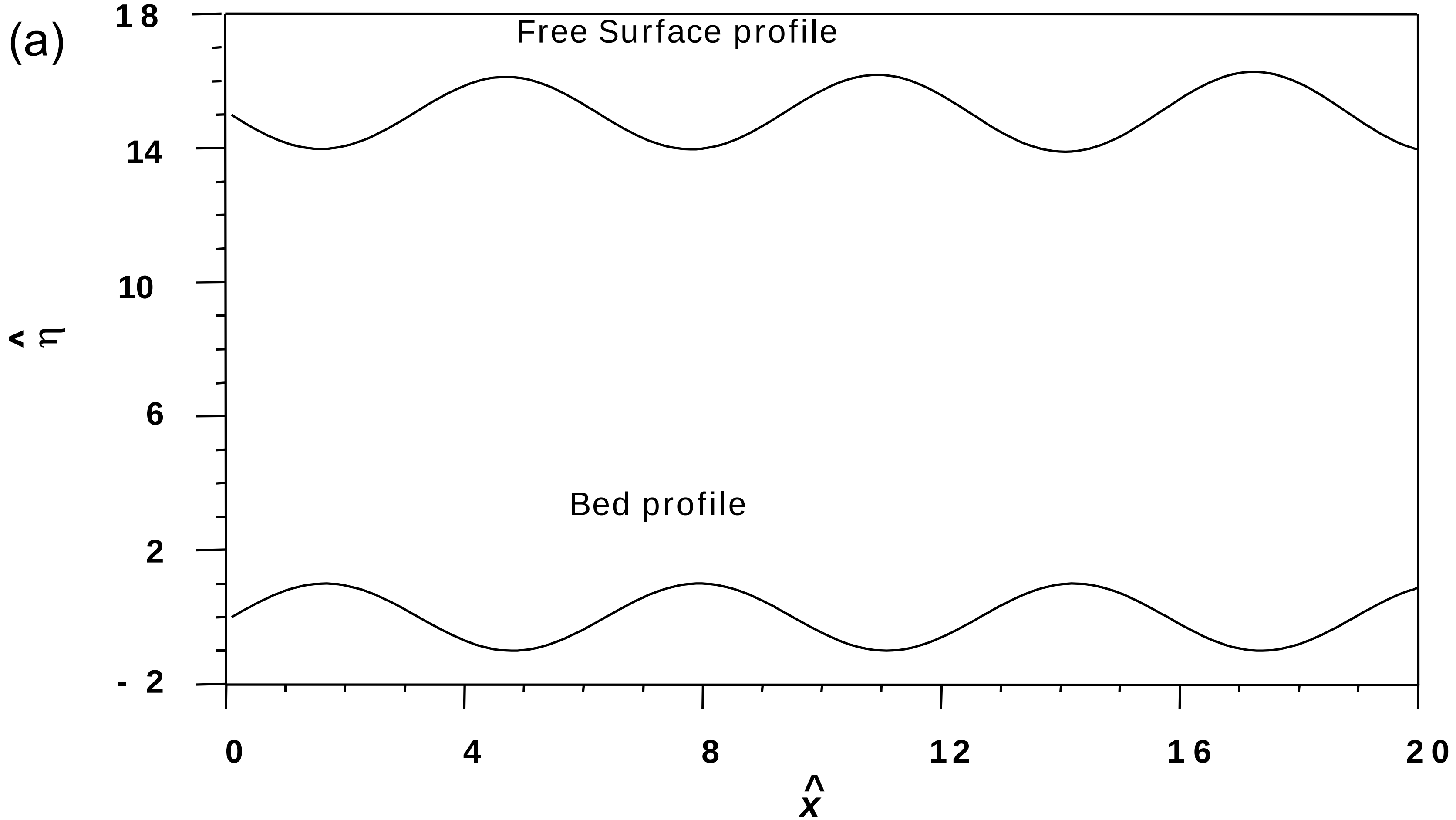


Figure 2(a) Nondimensional free surface profile for α = 0.1, β = 13, *F* = 0.2 and φ = 4×10-3

- The wavy free surface profile computed is shown in Figure 2(a). It is evident that the spatial lag is $\hat{x}$ = 3

- In another numerical experiment, the value of β was reduced to 9.5 keeping the other parameters unchanged. It means that the mean flow depth *D* is reduced.

- A satisfactory solution was obtained for the initial values of $\hat{\eta}$ = 0.8, $d\hat{\eta}/d\hat{x}$ = -1.41 and $d^2\hat{\eta}/d\hat{x}^2$ = –0.003 at the origin $\hat{x}$ = 0

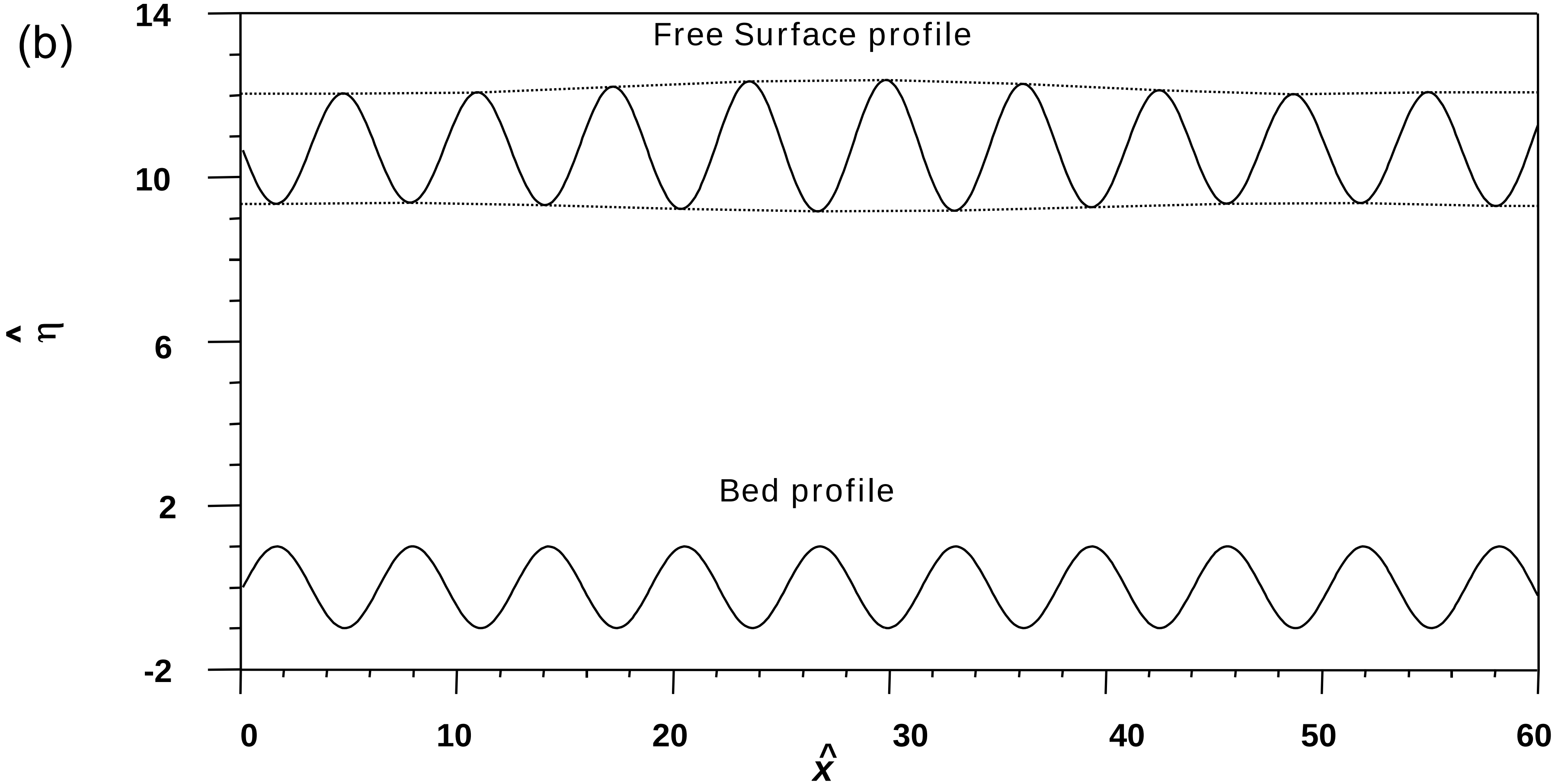


Figure 2 (b) nondimensional free surface profile for
α = 0.1, β = 9.5, F = 0.2 and φ = 4×10-3

- In Figure 2(b), the peaks of the waves definitely show periodic groups of waves in heaving motion.

Formation of Sand Waves

- The total-load transport q_T per unit time and width

$$q_T = q_B + \int_h^{\eta} \overline{u} c dy \tag{25}$$

where
q_B is the bed-load transport rate
c is the concentration of sand suspension

- The total-load satisfies the Exner's sediment continuity equation. It is

$$\frac{\partial q_T}{\partial x} = -(1-\rho_0)\frac{\partial h}{\partial t} - \frac{\partial}{\partial t}\int_h^{\eta} c(x,y,t)dy = -(1-\rho_0)\frac{\partial h}{\partial t} - \frac{\partial}{\partial t}[(\eta - h)C] \tag{26}$$

where ρ_0 is the porosity of bed sand and $C(x,t)$ is the depth-averaged concentration given by

$$C(x,t) = \frac{1}{\eta - h}\int_h^{\eta} c(x,y,t)dy \tag{27}$$

- Due to bed slope, modified bed-load q_B equation of Meyer-Peter and Müller

$$q_B = 8\sqrt{(s-1)gd^3}\left[\frac{\tau_0}{(s-1)\rho g d} - \mu\frac{\partial h}{\partial x} - 0.047\right]^{3/2} \tag{28}$$

where
s is the relative density of sand
d is the median sediment and
μ is the particle frictional coefficient (~ 0.1)

- The bed shear stress is obtained from Manning equation as $\tau_0 = \rho g n^2 U^2/(\eta - h)^{1/3}$. Sediment concentration c has an advection-diffusion equation of the type

$$\frac{Dc}{Dt} = w_s\frac{\partial c}{\partial y} + \left(\varepsilon_x\frac{\partial^2 c}{\partial x^2} + \varepsilon_y\frac{\partial^2 c}{\partial y^2}\right) \tag{29}$$

where
w_s is the terminal fall velocity of sand
ε_x is the turbulent diffusivity in x-direction
ε_v is the turbulent diffusivity in y-direction

- The diffusivities ε_x and ε_y are dependent on flow conditions. Thackston and Krenkel (1967) estimated ε_x as

$$\varepsilon_x = 7.25 u_\tau D (U / u_\tau)^{1/4} = 7.25 g^{3/8} n^{3/4} U D^{7/8} \tag{30}$$

- On the other hand, Lane and Kalinske (1941) estimated ε_y as

$$\varepsilon_y = u_\tau D / 15 = 0.066 g^{1/2} n U D^{5/6} \tag{31}$$

- From Eq. 26, the quantity of interest is the depth-averaged concentration *C*
- Thus, using Eq. 2 into Eq. 29 and integrating between limits *h* to η, yields

$$\int_h^\eta \frac{Dc}{Dt} dy = \frac{\partial}{\partial t}[(\eta - h)C] + \frac{\partial}{\partial x}\int_h^\eta \overline{u} c dy \tag{32}$$

- The integral of the right hand side in Eq. 29 equals

$$\left(w_s c + \varepsilon_y \frac{\partial c}{\partial y} \right)_h^{\eta} + \varepsilon_x \int_h^{\eta} \frac{\partial^2 c}{\partial x^2} dy \approx \varepsilon_x \frac{\partial^2}{\partial x^2}[(\eta - h)C] \qquad (33)$$

- In the above, the first term vanishes as there is no net vertical sediment flux across the extreme levels at $y = h$ and $y = \eta$. Eq. 29 thus leads to

$$\frac{\partial}{\partial t}[(\eta - h)C] + \frac{\partial}{\partial x}[(\eta - h)UC] = \varepsilon_x \frac{\partial^2}{\partial x^2}[(\eta - h)C] \qquad (34)$$

- Using Eqs. 25, 28 and 34 into the Exner equation, that is Eq. 26, yields

$$(1-\rho_0)\frac{\partial h}{\partial t} + \varepsilon_x \frac{\partial^2}{\partial x^2}[(\eta - h)C] + 12[(s-1)gd^3]^{0.5}\left[\frac{n^2U^2}{(\eta - h)^{1/3}(s-1)d} - \mu\frac{\partial h}{\partial x} - 0.047\right]^{0.5}$$

$$\times \left\{ \frac{n^2}{(\eta - h)^{1/3}(s-1)d}\left[2U\frac{\partial U}{\partial x} - \frac{1}{3}\cdot\frac{U^2}{\eta - h}\left(\frac{\partial \eta}{\partial x} - \frac{\partial h}{\partial x}\right)\right] - \mu\frac{\partial^2 h}{\partial x^2} \right\} = 0$$

(35)

- Eqs. 17, 21 (with σ = 1 and γ = 2/5), 34 and 35 constitute the equation of perturbed flow due to erosion of bed

- In Eq. 21, $\kappa(h) = \partial^2 h/\partial x^2$ and $\kappa(\eta) = \partial^2\eta/\partial x^2$ are taken.

- To investigate sand wave propagation, the above set of equations to the first order is linearized as

$$\frac{\partial \eta}{\partial t} - \frac{\partial h}{\partial t} + D\frac{\partial U}{\partial x} + U_m\left(\frac{\partial \eta}{\partial x} - \frac{\partial h}{\partial x}\right) = 0 \quad (36a)$$

$$\frac{\partial U}{\partial t} + U_m\frac{\partial U}{\partial x} + \frac{2}{5}DU_m^2\left(\frac{\partial^3 \eta}{\partial x^3} + \frac{7}{16}\cdot\frac{\partial^3 h}{\partial x^3}\right) + g\frac{\partial \eta}{\partial x} + \frac{gn^2U_m^2}{D^3} = 0 \quad (36b)$$

$$\frac{\partial C}{\partial t} + U_m\frac{\partial C}{\partial x} + C_0\frac{\partial U}{\partial x} = \varepsilon_x\frac{\partial^2 C}{\partial x^2} \quad (36c)$$

$$(1-\rho_0)\frac{\partial h}{\partial t} + \varepsilon_x D\frac{\partial^2 C}{\partial x^2} + G\left\{\frac{n^2U_m}{(s-1)dD^{1/3}}\left[2\frac{\partial U}{\partial x} - \frac{1}{3}\cdot\frac{U_m}{D}\left(\frac{\partial \eta}{\partial x} - \frac{\partial h}{\partial x}\right)\right] - \mu\frac{\partial^2 h}{\partial x^2}\right\} = 0 \quad (36d)$$

where G is $12[n^2gd^2U_m^2D^{-1/3} - 0.047(s-1)gd^3]^{0.5}$ and C_0 is the initial average concentration that may occur due to mean flow velocity U_m.

- If an exponential distribution of C_0 is assumed with ε_y given by Eq. 31, the average concentration C_0 is

$$C_0 = 4.853\times10^{-4}[(g^2 n^4 U_m^4)/(w_s^4 D^{2/3})] \qquad (37)$$

- For propagating wave type solution of the linear system of differential equations [Eqs. 36a - 36d with Eq. 37], the solution must be of the form

$$(\eta, h, U, C) = (\overline{E}, \overline{H}, \overline{U}, \overline{C})\exp(-\lambda t)\exp(ikx) \qquad (38)$$

where it is imperative that $Re(\lambda) > 0$ for bounded waves to propagate. Here, $Re(\lambda)$ denotes the real part of λ

- In this case, moving wavy bed-form is $h = \exp[-Re(\lambda)t]\exp\{i\,[kx - Im(\lambda)t]\}$ and the flow variables remain bounded for $t > 0$, where $Im(\lambda)$ denotes the imaginary part of λ. However, for $t \to \infty$, the return to zero value of h is physically inhibited by weakening erosion process, resulting in wavy bed-forms for all times

By substitution of Eq. 36b, noting that the constant term (last term) in Eq. 36b has no role in such a stable solution analysis; the following linear algebraic equations are therefore obtained:

$$(-\lambda + ikU_m)(\overline{E} - \overline{H}) + ikD\overline{U} = 0 \qquad (39a)$$

$$8i\left(5kg - 2k^3DU_m^2\right)\overline{E} - 7ik^3DU_m^2\overline{H} + 40(-\lambda + ikU_m)\overline{U} = 0 \qquad (39b)$$

$$ikC_0\overline{U} + (-\lambda + ikU_m + \varepsilon_x k^2)\overline{C} = 0 \qquad (39c)$$

$$-(1-\rho_0)\lambda\overline{H} - \varepsilon_x Dk^2\overline{C} + G\left\{\frac{ikn^2U_m}{(s-1)dD^{4/3}}\left[2D\overline{U} - \frac{1}{3}U_m(\overline{E} - \overline{H})\right] + \mu k^2\overline{H}\right\} = 0 \qquad (39d)$$

- Eliminating $\overline{E}$, $\overline{H}$, $\overline{U}$ and $\overline{C}$ from Eqs. 39a – 39d, one gets the quartic equation for λ:

$$(\lambda - ikU_m - \varepsilon_x k^2)\left\{\left[(1-\rho_0)\frac{\lambda}{G} - \mu k^2\right]\left[(\lambda - ikU_m)^2 + k^2D\left(-\frac{2}{5}Dk^2U_m^2 + g\right)\right] + \frac{n^2k^2U_m}{(s-1)dD^{1/3}}\right.$$

$$\left.\times\left(2\lambda - \frac{7}{3}ikU_m\right)\left(-\frac{23}{40}Dk^2U_m^2 + g\right)\right\} - \varepsilon_x\frac{k^4C_0D}{G}(\lambda - ikU_m)\left(-\frac{23}{40}Dk^2U_m^2 + g\right) = 0 \qquad (40)$$

- For formation of sand waves, the real parts of all the four roots must be positive

- In terms of nondimensional quantities as $X = \lambda(D/g)^{0.5}$, $\beta = kD$, $F_m = U_m/(gD)^{0.5}$, $\varphi_0 = D\varphi/[(s-1)d]$, $\varphi_A = 0.083/[(s-1)(F_m^2\varphi_0 - 0.047)]^{0.5}$, $\varepsilon = \varepsilon_x/(gD^3)^{0.5} = 7.25\varphi^{3/8}F_m$ and $C_0 = 4.853\times10^{-4}(F_m^2/\varphi^2)(u_\tau/w_s)^4$, Eq. 40 can then be written as a quartic equation of X. It is

$$(-X+i\beta F_m+\varepsilon\beta^2)\left\{\left[(1-\rho_0)\frac{X}{\varphi_A}-\mu\beta^2\right]\left[(X-i\beta F_m)^2+\beta^2\left(-\frac{2}{5}\beta^2F_m^2+1\right)\right]\right.$$
$$\left.+\varphi_0\beta^2F_m\left(2X-\frac{7}{3}i\beta F_m\right)\left(-\frac{23}{40}\beta^2F_m^2+1\right)\right\}+\varepsilon\frac{C_0\beta^2}{\varphi_A}(X-i\beta F_m)\left(-\frac{23}{40}\beta^2F_m^2+1\right)=0 \quad (41)$$

- Parameters are selected as $\varphi = 2.5\times10\text{-}3$, $\varphi_0 = 600\varphi$ and $u_\tau/w_s = 0.6$ for the computation of the four roots of X for different values of wave number β and Froude number F_m

- It transpires that all four roots have positive real parts when the points (β, F_m) in the β-F_m plot lie in a curved band forming a zone in which bed-forms remain unstable without becoming unbounded in

- This zone, where significant sediment transport takes place as bed-load and suspended-load, contains the experimental data of antidunes and standing waves, having higher values of F_m (> 0.8). The zone shrinks to an asymptotic critical line at $F_m = 0.177$, when β becomes large. Below this theoretical value no root of Eq. 41 exists and bed erosion is inhibited due to significant reduction of flow velocity

- If C_0 is set equal to zero, the transport process is due to bed-load only. In this case, the lower boundary of the unstable zone degenerates into the asymptotic line $F_m = 0.177$, that is the *lower limit for dune formation*

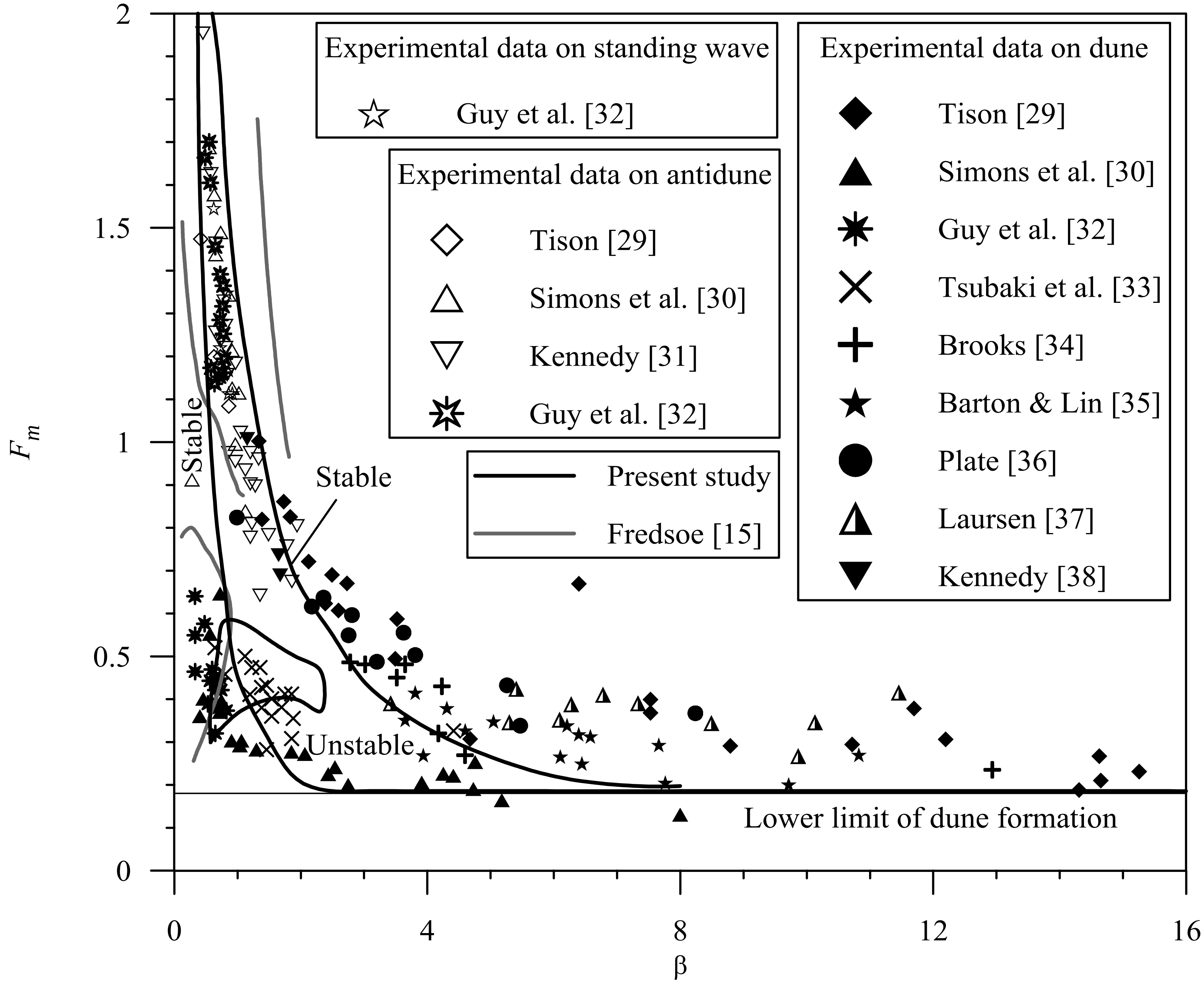


Figure 3 Stability of sand waves for φ = 2.5×10-3, φ_0 = 600φ and u_τ/w_s = 0.6

- Two basic equations obtained (Eqs. 17 and 20 or 21) can be regarded as the generalization of the Saint Venant equations of motion. In shear flow over a stable sinusoidal sand-bed, the free surface profile lags the bed profile, and when the flow depth decreases an accumulation of heaved waves in the free surface is formed

- In case of instability of a horizontal plane sand-bed, at higher Froude numbers F_m (> 0.8), the bed-forms remain unstable as standing waves and antidunes, while at lower F_m (with no suspended-load), the instability zone is extended to the lower limit of F_m = 0.177

Mechanics of Dunes

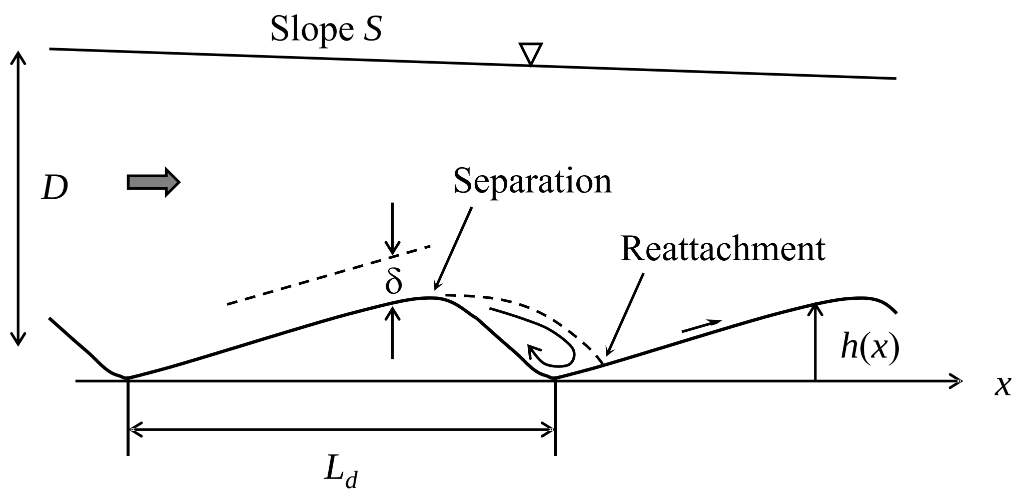


Fig. 5.2 Streamwise profiles of dunes

- Assuming that the bed waves are traveling at a constant velocity *a* without any change in shape
- Shape of the bed is described by an expression of the form

$$h = h(x - at) \tag{5.1}$$

where *h* = local height of the dune above an *x*-axis passing through the troughs

- Consider the bed-load of sediment q_B through two consecutive sections with unit spacing in the *x*-direction
- Net outflow of sediment is $\partial q_B / \partial x$, which is equal to the change in bed elevation when the correction for the porosity ρ_0 of the bed sediment is taken into account
- Change in the amount of sediment stored in suspension is ignored
- Sediment continuity equation (Exner equation) is

$$\frac{\partial q_B}{\partial x} = -(1-\rho_0)\frac{\partial h}{\partial t} \tag{5.2}$$

- If Eq. (5.1) is substituted into Eq. (5.2), they are satisfied by putting

$$q_B = q_B(x) = q_0 + a(1-\rho_0)h \tag{5.3}$$

where q_0 = a constant, interpreted as the value of q_B at h = 0, that is at the trough the bed-load is zero

- For small bed shear stress, q_0 becomes zero, because the sediment mainly moves as bed-load and so q_B becomes q_b
- The following relationship is obtained

$$q_b = a(1-\rho_0)h \tag{5.4}$$

- It is evident that the local intensity of bed-load transport is proportional to local height of bed above the plane through the troughs
- Bed shear stress at the dune surface must vary from zero at the trough to a maximum at the crest

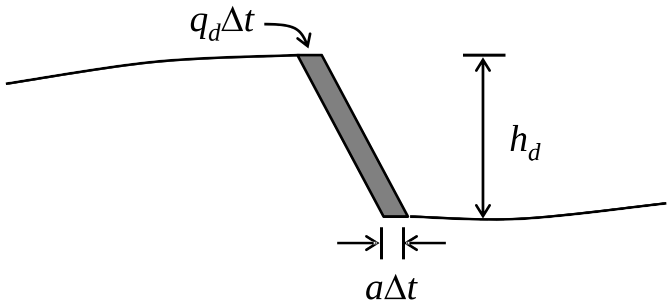


Fig. 5.3 Migration of dune front

- Migration velocity *a* of the dune

$$a = \frac{q_d}{(1-\rho_0)h_d} \tag{5.5}$$

where h_d = dune height, q_d = amount of sediment deposited

- From Eqs. (5.3) and (5.4), it is seen that for small bed shear stress, q_d becomes same as q_b
- The shape of the bed-form can be found from Eqs. (5.4) and (5.5), combining them as

$$\frac{h}{h_d} = \frac{q_b}{q_b\big|_{\text{top}}} \tag{5.6}$$

where $q_b\big|_{\text{top}}$ = sediment transport at the dune crest level where $h = h_d$

- The modified bed-load equation of Meyer-Peter due to bed slope is given by

$$q_b = 8\sqrt{\Delta g d^3}\left(\Theta - \frac{\Theta_c}{\tan\varphi}\cdot\frac{\partial h}{\partial x} - \Theta_c\right)^{3/2} \tag{5.7}$$

where d = representative particle diameter; g = acceleration due to gravity; $\Delta = s - 1$; s = relative density of sediment particles, that is ρ_s/ρ; ρ_s = mass density of sediment; ρ = mass density of fluid; Θ = local Shields parameter, $u_*^2/(\Delta g d)$; u_* = shear velocity; Θ_c = threshold Shields parameter; and φ = angle of repose of bed sediment

- Inserting Eq. (5.7) into Eq. (5.6) yields

$$\frac{1}{\tan\varphi}\cdot\frac{\partial h}{\partial x}+\left(\frac{\Theta_{\text{top}}}{\Theta_c}-1\right)\left(\frac{h}{h_d}\right)^{2/3}=\frac{\Theta}{\Theta_c}-1 \qquad (5.8)$$

- Due to the presence of the dunes, the local flow depth varies along the dune, which results in spatial changes to the depth-averaged flow velocity *U*, which varies along the dune

$$U(D+0.5h_d-h)=q \qquad (5.9)$$

where *q* = flow discharge per unit width

- Bed shear stress along the dune given by

$$\tau_0=\tau_{\text{top}}\,f\left(\frac{x}{h_d}\right)\left(\frac{U}{U_{\text{top}}}\right)^2 \qquad (5.10)$$

- Eqs. (5.9) and (5.10), yields

$$\Theta_c=\Theta_{\text{top}}\,f\left(\frac{x}{h_d}\right)\left(\frac{D-0.5h_d}{D+0.5h_d-h}\right)^2 \qquad (5.11)$$

Dune Height

- The dune height can be obtained from the geometrical consideration
- Combining Eqs. (5.1) and (5.5), gives

$$\frac{\partial h}{\partial t} = -a\frac{\partial h}{\partial x} = -\frac{q_d}{(1-\rho_0)h_d}\cdot\frac{\partial h}{\partial x} \quad (5.12)$$

- Combining Eqs. (5.2) and (5.12)

$$\frac{\partial q_B}{\partial x} = \frac{q_d}{h_d}\cdot\frac{\partial h}{\partial x} \quad (5.13)$$

- Eq. (5.13) can also be written

$$\frac{\partial \Phi_B}{\partial x} = \frac{\Phi_d}{h_d}\cdot\frac{\partial h}{\partial x} \quad (5.14)$$

where Φ_B = nondimensional sediment transport rate

- In steady flow, Φ_B is a function of Shields parameter Θ and hence

$$\frac{\partial \Phi_B}{\partial \Theta} \cdot \frac{\partial \Theta}{\partial x} = \frac{\Phi_d}{h_d} \cdot \frac{\partial h}{\partial x} \tag{5.15}$$

- At the dune crest, Eq. (5.11) is approximated by

$$\Theta = \Theta_{\text{top}} \left(\frac{D - 0.5 h_d}{D + 0.5 h_d - h} \right)^2 \tag{5.16}$$

- Θ_{top} appearing in Eq. (5.16) is the Shields parameter due to the skin friction, which can be related to the averaged skin friction Θ_{av}

$$\Theta_{\text{av}} = \Theta_{\text{top}} \left(\frac{D - 0.5 h_d}{D} \right)^2 \tag{5.17}$$

- Eq. (5.16) gives

$$\frac{\partial \Theta}{\partial x} = \frac{2\Theta_{\text{top}}}{D - 0.5 h_d} \cdot \frac{\partial h}{\partial x} \tag{5.18}$$

- Eq. (5.18) combines with Eq. (5.14)

$$\frac{h_d}{D} = \frac{2\Phi_d}{4\Theta \dfrac{d\Phi_B}{d\Theta} + \Phi_d}, \quad \Theta = \Theta_{\text{top}} \tag{5.19}$$

Dune Length

- The maximum bed shear stress is located around 16 times the dune height downstream from the former crest
- The maximum sediment transport rate, except for very small Shields parameters, occurs at the location of maximum dune height
- The equation of the dune length can be obtained from Eq. (5.6)

$$L_d = 16 h_d \tag{5.20}$$

- At higher bed shear stresses, where suspended sediment becomes the dominant transport mechanism, the situation becomes a little more complex because a spatial phase lag L_s is introduced between the location of the maximum bed shear stress and the location of the maximum suspended-load transport
- The maximum bed-load and suspended-load transport can be estimated

$$\frac{L_d}{h_d} = \frac{16 q_b + \left(\frac{L_s}{h_d} + 16 \right) q_s}{q_b + q_s} \tag{5.21}$$

where q_s = suspended-load transport rate

Phase Lag of Suspended Sediment

- The phase lag L_s is introduced because a sediment particle takes some time to settle after being picked up from the bed
- The magnitude of L_s can be estimated from the basic equation of sediment suspension

$$u\frac{\partial C}{\partial x} = w_{ss}\frac{\partial C}{\partial z} + \frac{\partial}{\partial z}\left(\varepsilon_s\frac{\partial C}{\partial z}\right) + \frac{\partial}{\partial x}\left(\varepsilon_s\frac{\partial C}{\partial x}\right) \qquad (5.22)$$

- In uniform flow, Eq. (5.22) can be solved giving

$$\frac{C}{C_{b0}} = \exp\left(-\frac{w_{ss}z}{\varepsilon_s}\right) \qquad (5.23)$$

where C_{b0} = nominal reference sediment concentration at the bed

- Concentration C_{b0} varies in the flow direction, for instance, due to spatial changes in the bed shear stress
- ε_s and u are assumed not to vary in the x-direction

- Because of the variation in C_{b0}, the vertical distribution of the suspended sediment deviates from the equilibrium profile given by Eq. (5.23)
- The vertical distribution of suspended sediment is still described by an exponential function introducing a steepness variable λ as

$$\frac{C}{C_{b0}} = \exp\left[-\frac{w_{ss}}{\varepsilon_s}(1+\lambda)z\right] \tag{5.24}$$

- Introducing Eq. (5.24) into the diffusion Eq. (5.22) and integrating over the flow depth

$$u\frac{\varepsilon_s}{w_{ss}} \cdot \frac{\partial}{\partial x}\left(\frac{C_{b0}}{1+\lambda}\right) = -C_{b0}w_{ss} + C_{b0}w_{ss}(1+\lambda) \tag{5.25}$$

- It is assumed that the sediment concentration vanishes towards the free surface, and the horizontal diffusion of sediment is neglected
- The variation in C_{b0} is taken to be a small perturbation, and Eq. (5.24) can then be linearized to give the following differential equation for the unknown parameter λ

$$\frac{d\lambda}{dx} + \frac{w_{ss}^2}{u\varepsilon_s}\lambda - \frac{1}{C_{b0}} \cdot \frac{\partial C_{b0}}{\partial x} = 0 \qquad (5.26)$$

- This equation can be solved for a given variation in C_{b0}
- As an example a periodic perturbation of C_{b0} is considered, giving a variation of

$$C_{b0} = C_0 + C_1 \sin(kx) \qquad (5.27)$$

- Introducing Eq. (5.27) into Eq. (5.25) and using that $C_0 >> C_1$

$$\frac{d\lambda}{dx} + \frac{\lambda}{L_s} - \frac{C_1 k}{C_{b0}} \cos(kx) = 0 \qquad (5.28)$$

- In Eq. (5.28), the length scale L_s is introduced

$$L_s = u\frac{\varepsilon_s}{w_{ss}^2} \qquad (5.29)$$

- The solution to Eq. (5.26) is

$$\lambda = \frac{C_1 k}{C_0} \cdot \frac{L_s}{1 + (kL_s)^2}[\cos(kx) + kL_s \sin(kx)] + C_2 \exp(-x / L_s) \qquad (5.30)$$

- The vanishing transient part can be ignored, that is $C_2 = 0$
- For long wavelengths of the perturbation, the nondimensional parameter kL_s is small, and Eq. (5.30) can be approximated

$$\lambda = \frac{C_1}{C_0} \cdot kL_s \cos(kx) \tag{5.31}$$

- The sediment transport rate q_s can be found

$$\begin{aligned} q_s &= \int_0^D Cudz = C_{b0} u \frac{\varepsilon_s}{w_{ss}} \cdot \frac{1}{1+\lambda} \approx C_{b0} u \frac{\varepsilon_s}{w_{ss}} (1-\lambda) \\ &= C_0 u \frac{\varepsilon_s}{w_{ss}} \left[1 + \frac{C_1}{C_0} \sin(kx) \right] \left[1 - \frac{C_1}{C_0} kL_s \cos(kx) \right] \\ &\approx C_0 u \frac{\varepsilon_s}{w_{ss}} \left\{ 1 + \frac{C_1}{C_0} [\sin(kx) - kL_s \cos(kx)] \right\} \\ &\approx C_0 u \frac{\varepsilon_s}{w_{ss}} \left\{ 1 + \frac{C_1}{C_0} \sin[k(x - L_s)] \right\} \end{aligned} \tag{5.32}$$

- For quasi-uniform condition, the suspended sediment transport rate

$$q_s = \int_0^D Cudz = C_{b0} u \frac{\varepsilon_s}{w_{ss}} = C_0 u \frac{\varepsilon_s}{w_{ss}} \left[1 + \frac{C_1}{C_0} \sin(kx) \right] \qquad (5.33)$$

- Comparing Eqs. (5.32) and (5.33), it is seen that the development in the concentration profile causes the sediment transport rate to have a phase lag L_s relative to the variation in the bed concentration

- The lag distance increases with a decrease in w_{ss} and with an increase in eddy viscosity ε_s or flow velocity u

- From Eqs. (5.23) and (5.29), the length scale L_s can be given

$$L_s = z_c \frac{u|_{z_c}}{w_{ss}} \qquad (5.34)$$

where z_c = height of the centroid of the concentration profile above the bed (= ε_s/w_{ss}); and $u|_{z_c}$ = average velocity at an elevation z_c

$$z_c = \frac{\int_0^D Czdz}{\int_0^D Cdz} \qquad (5.35)$$

Flow Resistance due to Dunes

- In presence of bed-forms, the resistance to the flow consists of two parts, one originating from the skin friction (or grain resistance) and other due to the expansion loss downstream of a dune crest

- The magnitude of the latter loss can be estimated from the Carnot equation

$$\Delta h'' = K \frac{(U_{cr} - U_{tr})^2}{2g} \qquad (5.36)$$

where U_{cr} = average velocity at the crest; U_{tr} = average velocity at the trough; and K = coefficient due to flow geometry

- The average velocities U_{cr} and U_{tr} are given by

$$U_{cr} = \frac{q}{D - 0.5 h_d} \qquad (5.37a)$$

$$U_{tr} = \frac{q}{D + 0.5 h_d} \qquad (5.37b)$$

where $q = UD$

- Therefore, Eq. (5.36) becomes

$$\Delta h'' \approx \alpha \frac{U^2}{2g}\left(\frac{h_d}{D}\right)^2 \tag{5.38}$$

- The total energy loss *J* per unit length in the streamwise direction is

$$J \approx J' + \frac{\Delta h''}{L_d} = J' + J'' \tag{5.39}$$

where *J*′ = gradient due to friction

- In a steady-uniform open channel flow, the total bed shear stress τ_0 is related to the energy gradient *J* by

$$\tau_0 = \rho g D J \tag{5.40}$$

- According to Eq. (5.39), τ_0 can be split into two parts

$$\tau_0 = \rho g D J' + \rho g D J'' = \tau' + \tau'' \tag{5.41}$$

where τ' = mean bed shear stress acting directly as a friction on the surface of the dune; and τ'' = form drag on the dune

- In nondimensional form, Eq. (5.41) can be expressed

$$\Theta = \Theta' + \Theta'' \qquad (5.42)$$

where Θ = Shields parameter, $\tau_0/(\Delta g d)^{0.5}$; $\Theta' = DJ'/(\Delta d)$; and $\Theta'' = 0.5\alpha(Uh_d)^2/(\Delta g D L_d)$

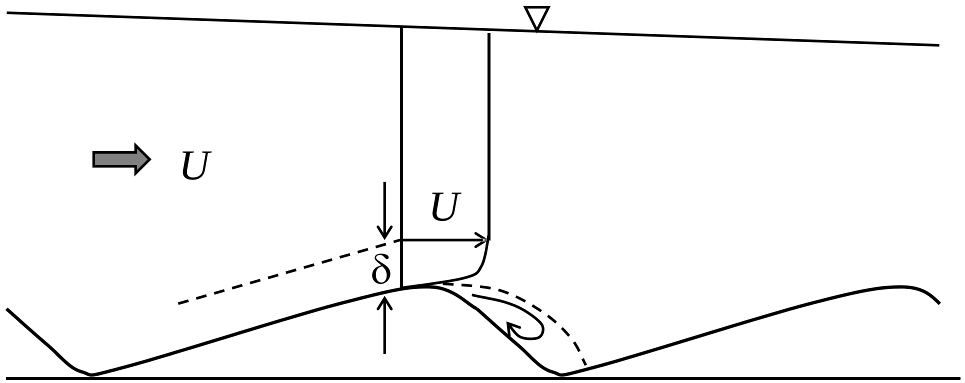


Fig. 5.4 Schematic of boundary layer developed along a dune

- To calculate J', an additional flow resistance equation for the skin friction is required
- Immediately downstream the dunes crest a wake like flow is formed producing large amount of turbulent energy
- This is dissipated into heat further downstream and thus causing the expansion loss

- At the end of the trough, a boundary layer with thickness δ is developed, where the velocity gradient is large, while the velocity distribution outside this layer is uniform

- **Engelund and Hansen** (1972) assumed that the upper flow and the boundary layer flow are independent of each other in the sense that no significant amount of energy is exchanged between them. Hence, the energy gradient of the boundary layer flow is equal to that of the upper layer and that of the total flow. It is

$$J = \frac{\tau_b' U'}{\rho g U' D'} = f' \frac{\rho U^2}{2\rho g D'} = f' \frac{U^2}{2gD'} \tag{5.43}$$

where U' = average flow velocity in the boundary layer; and f' = skin friction coefficient defined by

$$\tau_b' = f' \frac{\rho U'^2}{2} \tag{5.44}$$

- The expression of J is given by

$$J = f \frac{U^2}{2gD} \tag{5.45}$$

- Eqs. (5.43) and (5.45) produce

$$\frac{f'}{D'} = \frac{f}{D} \tag{5.46}$$

- Eqs. (5.45) and (5.46) gives

$$\sqrt{\frac{2}{f'}} = \frac{U}{\sqrt{gD'J}} \tag{5.47}$$

- The friction factor f' for the boundary layer is determined from the equation of Nikuradse

$$\sqrt{\frac{2}{f'}} = 6 + 2.5\ln\left(\frac{D'}{k_s}\right) \tag{5.48}$$

where k_s = equivalent sand roughness of Nikuradse

- As $f = f'$ and $D = D'$, the value of the constant in Eq. (5.48) is 6
- Using Eqs. (5.47) and (5.48)

$$\frac{U}{\sqrt{2gD'J}} = 6 + 2.5\ln\left(\frac{D'}{k_s}\right) \tag{5.49}$$

- The above equation was originally suggested by **Einstein** (1950), who obtained it as an analogy to his calculation of sidewall correction
- Combining Eqs. (5.43) and (5.45), yields the important expression

$$\tau'_b = \rho g D' J \tag{5.50}$$

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Thank You